More Green's Theorem

Saturday, April 17, 2021 9:01 AM

Recall

Def f is conservative if $f = \nabla F$ for some fcn F.

Technical point: If f is defined on a domain D (ie an open subset of $\mathbb{R}^2$ or $\mathbb{R}^3$) we say "f is conservative on D" if there's a fcn F on D s.t. $f = \nabla D$

Will see an example of a vector field F:

1. F is defined on D
2. F is not conservative on D
3. F is locally conservative on D

ie For each point, cannot find a single F that works for all of D.

Remark if D is connected, then any 2 potentials F_1 and F_2 for $\vec{f}$ must differ by a constant.

Definition $\vec{F}$ is path-independent (on D) if $\int_C \vec{f} \cdot d\vec{r}$ depends only on the endpoints of C (for $C \subseteq D$)

ie, if C_1, C_2 are curves in D w/ some endpoints (same direction), then

$$\int_{C_1} \vec{f} \cdot d\vec{r} = \int_{C_2} \vec{f} \cdot d\vec{r}$$

Thms (last time)

- conservative on D $\Rightarrow$ path independent on D
- $\vec{F}$ is path-ind. on D iff for every closed curve C in D, $\int_C \vec{f} \cdot d\vec{r} = 0$

Thm If $\vec{F}$ is path-ind on D, then $\vec{F}$ is conservative on D.

Proof sketch

Key if $\vec{F}$ path-ind on D, then for any two $P, Q \in D$, can define $\int_P^Q \vec{f} \cdot d\vec{r}$ w/o caring about which curve from P to Q we use.

Technical point true if D is connected. if not, thm still true but need to work seperately on each component of D.

Now choose $P_0 \in D$ and define

$$F(P) := \int_{P_0}^{P} \vec{f} \cdot d\vec{r}$$

Note: F must be path-ind for this to work

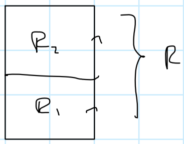
$\hookrightarrow$ can check that $\nabla F = \vec{f}$

$\blacksquare$ QED

Green's Theorem

→ see 4.3 [Co] ; notes linked

Last time



$$\int_R \vec{F}\cdot d\vec{r} = \int_{R_1} \vec{F}\cdot d\vec{r} + \int_{R_2} \vec{F}\cdot d\vec{r}$$

↑ default CCW

bc cancellation on the common edge intuitively, bc 2 gears going counterclockwise will grind against each other

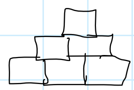
Generalization

If we have a grid:

R_{16}	17	18	19	20
11	12	13	14	15
6	7	8	9	10
R_1	R_2	R_3	R_4	R_5

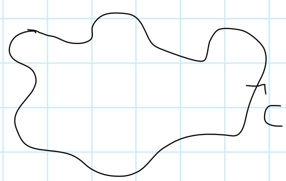
then the integral $\vec{F}\cdot d\vec{r}$ around the whole perimeter is $\sum_{i=1}^{20} \int_{R_i} \vec{F}\, d\vec{r}$

Note, the rectangles can be stacked in any way:



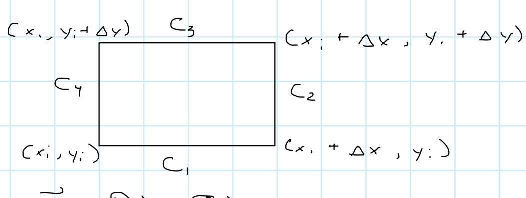
the integral around the outside perimeter is the sum of the integrals along each of the little rectangles

Idea: Suppose we have a closed curve like so:



compute $\int_C \vec{F}\cdot d\vec{r}$ by breaking the region bounded by C into lots of little rectangles, and approximating $\vec{F}$ on each rectangle then adding it all up.

Now let's integrate around a tiny rectangle



say $\vec{F} = P\vec{i} + Q\vec{j}$

guess $\int_{C_1}$ cancels $\int_{C_3}$

ACTUALLY from C_1 to C_3, $\vec{F}$ changes by $\Delta y \cdot \frac{\partial \vec{F}}{\partial x}$

$$\frac{\partial \vec{F}}{\partial x} = \frac{\partial P}{\partial x}\vec{i} + \frac{\partial Q}{\partial x}\vec{j}$$

Idea $\vec{F}(x, y_i+\Delta y) \approx F(x_i, y_i) + \Delta y \frac{\partial \vec{F}}{\partial y}(x, y_i)$

let parameterize C_1 & C_3

Param of C_1:

$(x, y) = (x_i + t\Delta x, y_i) \qquad 0 \le t \le 1$

$(x'(t), y'(t)) = (\Delta x, 0)$

Param of C_3:

$(x, y) = (x_i + \Delta x - t\Delta x, y_i + \Delta y) \qquad 0 \le t \le 1$

$(x'(t), y'(t)) = (-\Delta x, 0)$

Now

$$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_0^1 (P(x_i + t\Delta x, y_i), Q(x_i + t\Delta x, y_i)) \cdot (\Delta x, 0)\, dt$$

$$= \int_0^1 (P(x_i + t\Delta x, y_i)\Delta x\, dt$$

$$\int_{C_3} \vec{F} \cdot d\vec{r} = \int_0^1 F(x_i + \Delta x - t\Delta x, y_i + \Delta y) \cdot (-\Delta x, 0)\, dt$$

$$= -\int_0^1 P(x_i + \Delta x - t\Delta x, y_i + \Delta y)\Delta x\, dt$$

$$\approx -\int_0^1 \left[P(x_i + \Delta x - t\Delta x, y_i) + \Delta y \frac{\partial P}{\partial y}\right]\Delta x\, dt$$

$u = 1 - t$

$$= -\int_0^1 \left[P(x_i + u\Delta x, y_i) + \Delta y \frac{\partial P}{\partial y}\right]\Delta x\, du$$

$$= -\int_0^1 P(x_i + u\Delta x, y_i)\Delta x\, du - \int_0^1 \frac{\partial P}{\partial y} \Delta y \Delta x\, du$$

$$\int_{C_1 + C_3} \vec{F} \cdot d\vec{r} = \int_0^1 P(x_i + t\Delta x, y_i)\Delta x\, dt - \int_0^1 P(x_i + u\Delta x, y_i)\Delta x\, du - \int_0^1 \frac{\partial P}{\partial y} \Delta y \Delta x\, du$$

$$\approx -\int_0^1 \frac{\partial P}{\partial y} \Delta x \Delta y\, du$$

$$\approx -\frac{\partial P}{\partial y}(x_i, y_i)\Delta x \Delta y$$

Similarly

$$\int_{C_2 + C_4} \vec{F} \cdot d\vec{r} = \frac{\partial Q}{\partial x}\Delta x \Delta y$$

$$\Rightarrow \int_{\substack{\text{around} \\ \text{a little} \\ \text{rectangle} \\ \text{w/ sides} \\ \Delta x \,\&\, \Delta y}} \vec{F} \cdot d\vec{r} = \int_{C_1 + C_2 + C_3 + C_4} \vec{F}\, d\vec{r}$$

$\int$ around a little rectangle w/ sides Δx & Δy $\qquad \int_{C_1 + C_2 + C_3 + C_4}$

$$= \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \Delta x \Delta y$$

$$= \int_{\text{outer perimeter}} \vec{F} \cdot d\vec{r} = \sum_{\text{little rectangles}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \Delta x \Delta y$$

= Riemann sum for

$$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\, dy$$

where R is the region bounded by C